The Addition-Theorem for Elliptic Functions.

By William E. Story.

The form of the addition-theorem given below [(33)–(35)] is attributed by Clebsch* to Hermite,† whose note I have not seen, but the same result, presumably obtained by the same method, is given by Bertrand‡ and Koenigsberger§; of the two latter writers Koenigsberger alone investigates the effect of the equality of two or more of the arguments added, and neither considers the validity of the result when a certain intermediate equation (9) has equal roots. For this reason, and because the treatises cited are probably inaccessible to many American students, it seems allowable to present, even in a journal devoted to original research, the whole investigation in a brief but practically complete form.

Let $R(z)$ be a given cubic or quartic polynomial in z; we are concerned with the $2m-1$ (where m is any positive integer) integrals

$$(1)\qquad v_1=\int_{z_1'}^{z_1}\frac{dz}{\sqrt{R(z)}},\quad v_2=\int_{z_2'}^{z_2}\frac{dr}{\sqrt{R(z)}},\quad v_3=\int_{z_3'}^{z_3}\frac{dr}{\sqrt{R(z)}},\quad\ldots,\quad v_{2m-1}=\int_{z_{2m-1}'}^{z_{2m-1}}\frac{dz}{\sqrt{R(z)}},$$

whose upper limits $z_1, z_2, z_3, \ldots, z_{2m-1}$, and lower limits $z_1', z_2', z_3', \ldots, z_{2m-1}'$ have any given values, and the sign of $\sqrt{R(z)}$ for any value of z is determined by any convention consistent with continuity. It is to be observed that the number of integrals is odd. Now if $p(z)$ or p is an arbitrary polynomial in z of degree m, and $q(z)$ or q an arbitrary polynomial of degree not exceeding $m-2$, then $p-q\sqrt{R(z)}$ contains $m+1+m-1=2m$ arbitrary coefficients, which (*i. e.* whose ratios) may be so taken that

$$(2)\qquad p-q\sqrt{R(z)}=0$$

* Geometrie, I, p. 605, footnote.

† Note sur le calcul différentiel et le calcul intégral, in Lacroix: Calcul diff. et int., 6th ed., Paris, 1862, p. 68.

‡ Calcul intégral, pp. 578-583.

§ Elliptische Functionen, II, pp. 1-17.

for each of the $2m-1$ upper limits of the integrals (1); and if these upper limits are all different, this determination of the relative coefficients of (2) is unique, *i. e.* p and q are determined to a common factor *près*. Then (1) rationalized gives

(3) $$p^2 - q^2 R(z) = 0,$$

a rational equation of the degree $2m$ satisfied by the $2m-1$ given upper limits and therefore by one other value, say z_{2m}, which is thus completely determined by the $2m-1$ given values. Then

(4) $$p^2 - q^2 R(z) \equiv A(z-z_1)(z-z_2)(z-z_3)\ldots(z-z_{2m}),$$

where A is a constant (depending on the common arbitrary factor of p and q). If the given upper limits are not all different, suppose μ_1 of them equal to z_1, μ_2 equal to $z_2, \ldots, \mu_r$ equal to z_r, so that $\mu_1+\mu_2+\ldots+\mu_r = 2m-1$, then the coefficients of p and q can be determined, to a common factor *près*, in only one way, so that

for z_1, $p - q\sqrt{R(z)}$ and its first $\mu_1 - 1$ derivatives shall vanish,
" z_2, $p - q\sqrt{R(z)}$ " " $\mu_2 - 1$ " "
. . . .
. . . .
" z_r, $p - q\sqrt{R(z)}$ " " $\mu_r - 1$ " " .

Now it is easily seen that if, for any value of z, $p - q\sqrt{R(z)}$ and its first $\mu - 1$ derivatives vanish, then also will $p^2 - q^2R(z)$ and its first $\mu - 1$ derivatives vanish for the same value of z; hence

(5) $$p^2 - q^2R(z) \equiv A(z-z_1)^{\mu_1}(z-z_2)^{\mu_2}\ldots(z-z_r)^{\mu_r}(z-z_{2m}),$$

where z_{2m} is a value determined by the $2m-1$ given upper limits, and A is a constant. Similarly if $p_1(z)$ or p_1 is a polynomial of degree m in z, and $q_1(z)$ or q_1 a polynomial in z of degree not exceeding $m-2$, the coefficients of p_1 and q_1 can be determined in one way only, to a constant factor *près*, so that

(6) $$p_1 - q_1\sqrt{R(z)} = 0$$

for each of the $2m-1$ lower limits of the integral (1), if these lower limits are all different; if any lower limit z' occurs μ times, then p_1 and q_1 are to be so determined that $p_1 - q_1\sqrt{R(z)}$ and its first $\mu - 1$ derivatives shall vanish for $z = z'$; and the coefficients of p_1 and q_1 so taken determine a value z'_{2m} so connected with the $2m-1$ given lower limits that

(7) $$p_1^2 - q_1^2 R(z) \equiv B(z-z'_1)(z-z'_2)(z-z'_3)\ldots(z-z'_{2m}),$$

where B is a constant. The value of z_{2m} satisfies (2) as well as (3), and z'_{2m} satisfies

(6), viz. the sign of $\sqrt{R(z)}$ is to be so taken for z_{2m} and z'_{2m} that these equations shall be satisfied. The values z_{2m} and z'_{2m} determine another integral

$$(8)\qquad v_{2m}=\int_{z'_{2m}}^{z_{2m}}\frac{dz}{\sqrt{R(z)}},$$

whose relation to the $2m-1$ given integrals we have to investigate.

Let a new variable λ be introduced, and for any given value of λ let $\zeta_1, \zeta_2, \zeta_3, \ldots \zeta_{2m}$ be the $2m$ values of z which satisfy the equation

$$(8)\qquad (p+\lambda p_1)-(q+\lambda q_1)\sqrt{R(z)}=0,\ \text{or}$$

$$(9)\qquad (p+\lambda p_1)^2-(q+\lambda q_1)^2R(z)=0,$$

so that

$$(10)\quad (p+\lambda p_1)^2-(q+\lambda q_1)^2R(z)\equiv\psi(z)\equiv C(z-\zeta_1)(z-\zeta_2)(z-\zeta_3)\ldots(z-\zeta_{2m}).$$

If λ varies continuously from 0 to ∞, the roots of (10) vary continuously from $z_1, z_2, z_3, \ldots, z_{2m}$ to $z'_1, z'_2, z'_3, \ldots, z_{2m}$. It is of no consequence if any upper limit does not pass into the lower limit of the same integral by this continuous variation of λ. Since $\zeta_1, \zeta_2, \zeta_3, \ldots \zeta_{2m}$ and C are functions of λ defined by (10), *i. e.* this equation is an identity, we may differentiate it with respect to λ and obtain

$$(11)\left\{\begin{aligned}&2(p+\lambda p_1)p_1-2(q+\lambda q_1)q_1R(z)\equiv\frac{2(p+\lambda p_1)}{q+\lambda q_1}\left[(q_1+\lambda q_1)p_1-\frac{(q+\lambda q_1)^2}{p+\lambda p_1}R(z)\right]\\&\qquad\equiv\psi(z)\left[\frac{\frac{\partial\zeta_1}{\partial\lambda}}{\zeta_1-z}+\frac{\frac{\partial\zeta_2}{\partial\lambda}}{\zeta_2-z}+\ldots+\frac{\frac{\partial\zeta_{2m}}{\partial\lambda}}{\zeta_{2m}-z}+\frac{\frac{\partial C}{\partial\lambda}}{C}\right];\end{aligned}\right.$$

but
$$(q+\lambda q_1)^2R(z)\equiv(p+\lambda p_1)^2-\psi(z),$$

$$(q+\lambda q_1)p_1-\frac{(q+\lambda q_1)^2q_1}{p+\lambda p_1}R(z)\equiv(qp_1-pq_1)+\frac{q_1\psi(z)}{p+\lambda p_1},$$

i. e. (11) may be written

$$(12)\ 2\frac{(p+\lambda p_1)}{q+\lambda q_1}(qp_1-pq_1)+2\frac{q_1\psi(z)}{q+\lambda q_1}\equiv\psi(z)\left[\frac{\frac{\partial\zeta_1}{\partial\lambda}}{\zeta_1-z}+\frac{\frac{\partial\zeta_2}{\partial\lambda}}{\zeta_2-z}+\ldots+\frac{\frac{\partial\zeta_{2m}}{\partial\lambda}}{\zeta_{2m}-z}+\frac{\frac{\partial C}{\partial\lambda}}{C}\right].$$

If α represents any one of the numbers $1, 2, 3, \ldots 2m$,

$$\psi(\zeta_\alpha)=0,\ \left(\frac{\psi(z)}{\zeta_\alpha-z}\right)_{z=\zeta_\alpha}=-\frac{\partial\psi(\zeta_\alpha)}{\partial\zeta_\alpha}=-\psi'(\zeta_\alpha),$$

and by (8)
$$\frac{p(\zeta_\alpha)+\lambda p_1(\zeta_\alpha)}{q(\zeta_\alpha)+\lambda q_1(\zeta_\alpha)}=\sqrt{R(\zeta_\alpha)};$$

and (12) gives

$$2\sqrt{R(\zeta_\alpha)}\,[q(\zeta_\alpha)p_1(\zeta_\alpha)-p(\zeta_\alpha)q_1(\zeta_\alpha)]=-\psi'(\zeta_\alpha)\frac{\partial\zeta_\alpha}{\partial\lambda},$$

i. e.

$$(13)\qquad \frac{\dfrac{\partial\zeta_\alpha}{\partial\lambda}}{\sqrt{R(\zeta_\alpha)}} = -2\frac{q(\zeta_\alpha)p_1(\zeta_\alpha)-p(\zeta_\alpha)q_1(\zeta_\alpha)}{\psi'(\zeta_\alpha)},$$

and hence

$$(14)\qquad \sum_1^{2m}{}_\alpha \frac{\dfrac{\partial\zeta_\alpha}{\partial\lambda}}{\sqrt{R(\zeta_\alpha)}} = -2\sum_1^{2m}{}_\alpha \frac{q(\zeta_\alpha)p_1(\zeta_\alpha)-p(\zeta_\alpha)q_1(\zeta_\alpha)}{\psi'(\zeta_\alpha)} = 0,$$

by a well-known theorem of rational fractions, since $qp_1 - pq_1$ is of degree not higher than $2m-2$, and $\psi(z)$ is of degree m (see Todhunter's Theory of Equations, p. 325, example 13). If ζ_α is a multiple root of (9), say of order $\mu_\alpha \overline{>} 2$, then $\dfrac{\psi(z)}{\zeta_\alpha - z}$ contains $(z-\zeta_\alpha)^{\mu_\alpha-1}$ and (12) shows that $qp_1 - pq_1$ contains $(z-\zeta_\alpha)^{\mu_\alpha-1}$, since $\left(\dfrac{p+\lambda p_1}{q+\lambda q_1}\right)_{z=\zeta_\alpha} = \sqrt{R(\zeta_\alpha)}$ does not in general vanish. But differentiating (12) $\mu_\alpha - 1$ times and putting $z = \zeta_\alpha$ we obtain

$$2\frac{\partial^{\mu_\alpha-1}}{\partial\zeta_\alpha^{\mu_\alpha-1}}\Big[\{q(\zeta_\alpha)p_1(\zeta_\alpha)-p(\zeta_\alpha)q(\zeta_\alpha)\}\sqrt{R(\zeta_\alpha)}\Big] = -\frac{\partial^{\mu_\alpha}\psi(\zeta_\alpha)}{\partial\zeta_\alpha^{\mu_\alpha}}\frac{\partial\zeta_\alpha}{\partial\lambda}$$
$$= 2\sqrt{R(\zeta_\alpha)}\frac{\partial^{\mu_\alpha-1}}{\partial\zeta_\alpha^{\mu_\alpha-1}}[q(\zeta_\alpha)p_1(\zeta_\alpha)-p(\zeta_\alpha)q_1(\zeta_\alpha)],$$

i. e.

$$(15)\quad \frac{\dfrac{\partial\zeta_\alpha}{\partial\lambda}}{\sqrt{R(\zeta_\alpha)}} = -2\frac{\dfrac{\partial^{\mu_\alpha-1}}{\partial\zeta_\alpha^{\mu_\alpha-1}}[q(\zeta_\alpha)p_1(\zeta_\alpha)-p(\zeta_\alpha)q_1(\zeta_\alpha)]}{\dfrac{\partial^{\mu_\alpha}\psi(\zeta_\alpha)}{\partial\zeta_\alpha^{\mu_\alpha}}}$$
$$= -2\frac{\partial^{\mu_\alpha-1}}{\partial\zeta_\alpha^{\mu_\alpha-1}}\left[\frac{q(\zeta_\alpha)p_1(\zeta_\alpha)-p(\zeta_\alpha)q_1(\zeta_\alpha)}{\dfrac{\partial^{\mu_\alpha}\psi(\zeta_\alpha)}{\partial\zeta_\alpha^{\mu_\alpha}}}\right].$$

Now Jacobi has shown* that if $\phi(x)$ and $\psi(x)$ are polynomials in x such that the degree of $\psi(x)$ exceeds that of $\phi(x)$ by at least one unit, and if a is a multiple root of order μ of $\psi(z) = 0$, so that $\psi(x) \equiv (x-a)^\mu\psi_1(x)$, then the μ terms in the development of $\dfrac{\varphi(x)}{\psi(x)}$ in partial fractions whose denominators have become equal to a are replaced by

$$\frac{\dfrac{\varphi(a)}{\psi_1(a)}}{(x-a)^\mu} + \frac{\dfrac{\partial}{\partial a}\left(\dfrac{\varphi(a)}{\psi_1(a)}\right)}{(x-a)^{\mu-1}} + \frac{\dfrac{\partial^2}{\partial a^2}\left(\dfrac{\varphi(a)}{\psi_1(a)}\right)}{2!\,(x-a)^{\mu-2}} + \dots + \frac{\dfrac{\partial^{\mu-1}}{\partial a^{\mu-1}}\left(\dfrac{\varphi(a)}{\psi_1(a)}\right)}{(\mu-1)!\,(x-a)}$$
$$= \frac{1}{(\mu-1)!}\frac{\partial^{\mu-1}}{\partial a^{\mu-1}}\left(\frac{\varphi(a)}{(x-a)\psi_1(a)}\right) = \mu\frac{\partial^{\mu-1}}{\partial a^{\mu-1}}\left(\frac{\varphi(a)}{(x-a)\dfrac{\partial^\mu\psi(a)}{\partial a^\mu}}\right),$$

* Disquisitiones analyticae de fractionibus simplicibus. Inaugural dissertation. Werke, herausgegeben von Weierstrass, Vol. III, p. 11.

so that

$$(16)\qquad \frac{\varphi(x)}{\psi(x)} \equiv \sum \mu \frac{\partial^{\mu-1}}{\partial a^{\mu-1}} \left(\frac{\varphi(a)}{(x-a)\dfrac{\partial^\mu \psi(a)}{\partial a^\mu}} \right),$$

where the summation extends to every root a of $\psi(x) = 0$ and its index of multiplicity μ. Writing $xf(x)$ instead of $\phi(x)$, where $f(x)$ is any polynomial whose degree falls short of that of $\psi(x)$ by at least two units, we obtain from (16)

$$\frac{xf(x)}{\psi(x)} \equiv \sum \mu \frac{\partial^{\mu-1}}{\partial a^{\mu-1}} \left(\frac{af(a)}{(x-a)\dfrac{\partial^\mu \psi(a)}{\partial a^\mu}} \right),$$

and hence, for $x = 0$,

$$(17)\qquad 0 = \sum \mu \frac{\partial^{\mu-1}}{\partial a^{\mu-1}} \left(\frac{af(a)}{-a\dfrac{\partial^\mu \psi(a)}{\partial a^\mu}} \right) = -\sum \mu \frac{\partial^{\mu-1}}{\partial a^{\mu-1}} \left(\frac{f(a)}{\dfrac{\partial^\mu \psi(a)}{\partial a^\mu}} \right),$$

which is the generalization of the formula cited in connection with (14). This formula applied to (15) gives

$$(18)\qquad \sum \mu_a \frac{\dfrac{\partial \zeta_a}{\partial \lambda}}{\sqrt{R(\zeta_a)}} = 0,$$

where the summation extends to every root ζ_a of (9) and its index of multiplicity μ_a. But (18) is only what (14) becomes when μ_a values ζ_a are equal. We have then

$$\sum_1^{2m}{}_a \frac{\dfrac{\partial \zeta_a}{\partial \lambda}}{\sqrt{R(\zeta_a)}} = 0,$$

where the summation extends to $\zeta_1, \zeta_2, \zeta_3, \ldots \zeta_{2m}$, whether these are all different or not. Hence

$$(19)\qquad \sum_1^{2m}{}_a \int_\infty^0 \frac{\dfrac{\partial \zeta_a}{\partial \lambda}}{\sqrt{R(\zeta_a)}} d\lambda = \sum_1^{2m}{}_a \int_{z'_a}^{z_a} \frac{d\zeta_a}{\sqrt{R(\zeta_a)}} = \sum_1^{2m}{}_a v_a = 0;$$

i. e.

$$(20)\qquad v_{2m} = -(v_1 + v_2 + v_3 + \ldots + v_{2m-1}),$$

which is the relation between v_{2m} and the $2m-1$ given integrals corresponding to the relations above mentioned between z_{2m} and the $2m-1$ given upper limits and between z'_{2m} and the $2m-1$ given lower limits. The latter relations may be put into a simpler form (evidently this is only one of $2m$ analogous relations), viz. (4) and (7) give, if the constants A and B be taken equal to unity,

$$p^2(0) - q^2(0)R(0) = z_1 z_2 z_3 \ldots z_{2m}, \quad p_1^2(0) - q_1^2(0)R(0) = z'_1 z'_2 z'_3 \ldots z'_{2m},$$

i. e.

(21) $$z_{2m}=\frac{p^2(0)-q^2(0)\,R(0)}{z_1 z_2 z_3\ldots z_{2m-1}},\quad z'_{2m}=\frac{p_1^2(0)-q_1^2(0)\,R(0)}{z'_1 z'_2 z'_3\ldots z'_{2m-1}}.$$

In particular if

(22) $$R(z)\equiv z(1-z)(1-k^2z),\quad z=x^2,\quad x=\operatorname{sn}(u,k),$$

then

(23) $$\begin{cases}\sqrt{R(z)}\equiv \operatorname{sn}u\operatorname{cn}u\operatorname{dn}u,\quad R(0)=0,\quad R(1)=0,\quad R\left(\frac{1}{k^2}\right)=0,\\ \displaystyle\int_0^z\frac{dr}{\sqrt{R(z)}}=2\int_0^x\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=2u,\end{cases}$$

and if
$$z_1=\operatorname{sn}^2u_1,\ z_2=\operatorname{sn}^2u_2,\ z_3=\operatorname{sn}^2u_3,\ \ldots\ z_{2m}=\operatorname{sn}^2u_{2m},$$
$$z'_1=\operatorname{sn}^2u'_1,\ z'_2=\operatorname{sn}^2u'_2,\ z'_3=\operatorname{sn}^2u'_3,\ \ldots\ z'_{2m}=\operatorname{sn}^2u'_{2m},$$

then
$$\int_{z'_\alpha}^{z_\alpha}\frac{dz}{\sqrt{R(z)}}=2(u_\alpha-u'_\alpha),$$

and (19) and (21) become

(24) $$\sum_1^{2m}{}_\alpha\, u_\alpha=\sum_1^{2m}{}_\alpha\, u'_\alpha,$$

$$z_{2m}=\frac{p^2(0)}{z_1z_2z_3\ldots z_{2m-1}},\quad z'_{2m}=\frac{p_1^2(0)}{z'_1z'_2z'_3\ldots z'_{2m-1}},$$

i. e.

(25) $$\operatorname{sn}u_{2m}=\pm\frac{p(0)}{\operatorname{sn}u_1\operatorname{sn}u_2\operatorname{sn}u_3\ldots\operatorname{sn}u_{2m-1}}.$$

Assuming the same particular form of $R(z)$ and the same values of A and B in (4) and (7), we obtain from the two latter equations

$$p^2(1)-q^2(1)\,R(1)=p^2(1)=(1-z_1)(1-z_2)(1-z_3)\ \ldots\ (1-z_{2m})$$
$$=\operatorname{cn}^2u_1\operatorname{cn}^2u_2\operatorname{cn}^2u_3\ldots\operatorname{cn}^2u_{2m},$$

$$p^2\left(\frac{1}{k^2}\right)-q^2\left(\frac{1}{k^2}\right)R\left(\frac{1}{k^2}\right)=p^2\left(\frac{1}{k^2}\right)=\frac{1}{k^{4m}}(1-k^2z_1)(1-k^2z_2)(1-k^2z_3)\ldots(1-k^2z_{2m})$$
$$=\frac{1}{k^{4m}}\operatorname{dn}^2u_1\operatorname{dn}^2u_2\operatorname{dn}^2u_3\ldots\operatorname{dn}^2u_{2m};$$

and hence follow

(26) $$\operatorname{cn}(u_{2m})=\pm\frac{p(1)}{\operatorname{cn}u_1\operatorname{cn}u_2\operatorname{cn}u_3\ldots\operatorname{cn}u_{2m-1}},$$

(27) $$\operatorname{dn}(u_{2m})=\pm\frac{k^{2m}p\left(\frac{1}{k^2}\right)}{\operatorname{dn}u_1\operatorname{dn}u_2\operatorname{dn}u_3\ldots\operatorname{dn}u_{2m-1}}.$$

It is to be observed that the signs of the right members of (25), (26) and (27) have yet to be determined.

If we assume still further

(28) $$z'_1 = z'_2 = z'_3 = \ldots = z'_{2m-1} = 0,$$

we have

(29) $$v'_1 = v'_2 = v'_3 = \ldots = v'_{2m-1} = 0,$$

and, by (7),

(30) $$p_1^2 - q_1^2 R(z) \equiv z^{2m-1}(z - z'_{2m});$$

now the degree of the lowest term in p_1^2 is even and that of the lowest term in $q_1^2 R(z)$ is odd, so that these terms cannot cancel each other, and the degree of the second (the apparently lowest) term in $z^{2m-1}(z - z'_{2m})$ is $2m - 1$; this second term cannot arise from p_1^2 since it is of odd degree, and it cannot arise from $q_1^2 R(z)$ since the lowest term in $q_1^2 R(z)$ is of degree not higher than $2m - 4 + 1 = 2m - 3$; therefore it does not exist, *i. e.*

(31) $$z'_{2m} = 0, \quad v'_{2m} = 0,$$

and (24) becomes

$$\sum_1^{2m} {}^{\alpha} u_\alpha = 0,$$

i. e.

(32) $$u_{2m} = -(u_1 + u_2 + u_3 + \ldots + u_{2m-1}).$$

Equations (25), (26) and (27) may then be written

(33) $$\operatorname{sn}(u_1 + u_2 + u_3 + \ldots + u_{2m-1}) = \pm \frac{p(0)}{\operatorname{sn} u_1 \operatorname{sn} u_2 \operatorname{sn} u_3 \ldots \operatorname{sn} u_{2m-1}},$$

(34) $$\operatorname{cn}(u_1 + u_2 + u_3 + \ldots + u_{2m-1}) = \pm \frac{p(1)}{\operatorname{cn} u_1 \operatorname{cn} u_2 \operatorname{cn} u_3 \ldots \operatorname{cn} u_{2m-1}},$$

(35) $$\operatorname{dn}(u_1 + u_2 + u_3 + \ldots + u_{2m-1}) = \pm \frac{k^{2m} p\left(\frac{1}{k^2}\right)}{\operatorname{dn} u_1 \operatorname{dn} u_2 \operatorname{dn} u_3 \ldots \operatorname{dn} u_{2m-1}},$$

where the coefficients of p are to be determined by the conditions derived from (2), namely

(36) $$\begin{cases} p(\operatorname{sn}^2 u_1) - q(\operatorname{sn}^2 u_1) . \operatorname{sn} u_1 \operatorname{cn} u_1 \operatorname{dn} u_1 = 0, \\ p(\operatorname{sn}^2 u_2) - q(\operatorname{sn}^2 u_2) . \operatorname{sn} u_2 \operatorname{cn} u_2 \operatorname{dn} u_2 = 0, \\ p(\operatorname{sn}^2 u_3) - q(\operatorname{sn}^2 u_3) . \operatorname{sn} u_3 \operatorname{cn} u_3 \operatorname{dn} u_3 = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ p(\operatorname{sn}^2 u_{2m-1}) - q(\operatorname{sn}^2 u_{2m-1}) . \operatorname{sn} u_{2m-1} \operatorname{cn} u_{2m-1} \operatorname{dn} u_{2m-1} = 0. \end{cases}$$

If we put for convenience

$$\operatorname{sn} u = s,\ \operatorname{cn} u = c,\ \operatorname{dn} u = d,\ \operatorname{sn} u_\alpha = s_\alpha,\ \operatorname{cn} u_\alpha = c_\alpha,\ \operatorname{dn} u_\alpha = d_\alpha,$$

we may write

$$(37)\qquad \begin{cases} p(z) - q(z)\sqrt{R(z)} \equiv s^{2m} + a_1 s^{2m-2} + a_2 s^{2m-4} + \ldots + a_m \\ \qquad + (b_2 s^{2m-4} + b_3 s^{2m-6} + \ldots + b_m)\, scd, \end{cases}$$

which must then vanish for $s = s_1, s_2, s_3, \ldots, s_{2m-1}$, *i. e.* $a_1, a_2, \ldots, a_m, b_2, \ldots, b_m$ are determined by the linear equations

$$(38)\qquad \begin{cases} 0 = s_1^{2m} + a_1 s_1^{2m-2} + a_2 s_1^{2m-4} + \ldots + a_m + (b_2 s_1^{2m-4} + b_3 s_1^{2m-6} + \ldots + b_m)\, s_1 c_1 d_1, \\ 0 = s_2^{2m} + a_1 s_2^{2m-2} + a_2 s_2^{2m-4} + \ldots + a_m + (b_2 s_2^{2m-4} + b_3 s_2^{2m-6} + \ldots + b_m)\, s_2 c_2 d_2, \\ 0 = s_3^{2m} + a_1 s_3^{2m-2} + a_2 s_3^{2m-4} + \ldots + a_m + (b_2 s_3^{2m-4} + b_3 s_3^{2m-6} + \ldots + b_m)\, s_3 c_3 d_3, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = s_{2m-1}^{2m} + a_1 s_{2m-1}^{2m-2} + a_2 s_{2m-1}^{2m-4} + \ldots + a_m \\ \qquad + (b_2 s_{2m-1}^{2m-4} + b_3 s_{2m-1}^{2m-6} + \ldots + b_m)\, s_{2m-1} c_{2m-1} d_{2m-1}, \end{cases}$$

and we have

$$(39)\qquad \begin{cases} p(0) = a_m, \\ p(1) = 1 + a_1 + a_2 + a_3 + \ldots + a_m, \\ p\left(\dfrac{1}{k^2}\right) = \dfrac{1}{k^{2m}}(1 + a_1 k^2 + a_2 k^4 + a_3 k^6 + \ldots + a_m k^{2m}). \end{cases}$$

Write for convenience

$$\begin{vmatrix} s_1^{2m-2}, & s_1^{2m-4}, & \ldots, & s_1^2, & 1, & s_1^{2m-3} c_1 d_1, & s_1^{2m-5} c_1 d_1, & \ldots, & s_1 c_1 d_1 \\ s_2^{2m-2}, & s_2^{2m-4}, & \ldots, & s_2^2, & 1, & s_2^{2m-3} c_2 d_2, & s_2^{2m-5} c_2 d_2, & \ldots, & s_2 c_2 d_2 \\ s_3^{2m-2}, & s_3^{2m-4}, & \ldots, & s_3^2, & 1, & s_3^{2m-3} c_3 d_3, & s_3^{2m-5} c_3 d_3, & \ldots, & s_3 c_3 d_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{2m-1}^{2m-2}, & s_{2m-1}^{2m-4}, & \ldots, & s_{2m-1}^2, & 1, & s_{2m-1}^{2m-3} c_{2m-1} d_{2m-1}, & s_{2m-1}^{2m-5} c_{2m-1} d_{2m-1}, & \ldots, & s_{2m-1} c_{2m-1} d_{2m-1} \end{vmatrix}$$

$$= \begin{vmatrix} s_\alpha^{2m-2},\ s_\alpha^{2m-4},\ \ldots,\ s_\alpha^2,\ 1,\ s_\alpha^{2m-3} c_\alpha d_\alpha,\ s_\alpha^{2m-5} c_\alpha d_\alpha,\ \ldots,\ s_\alpha c_\alpha d_\alpha \\ \alpha = 1,\ 2,\ 3,\ \ldots,\ 2m-1 \end{vmatrix},$$

and similarly for any determinant whose rows differ only in the suffixes involved in them; then (38) and (39) give

$$p(0) \begin{vmatrix} s_\alpha^{2m-2},\ s_\alpha^{2m-4},\ \ldots,\ s_\alpha^2,\ 1,\ s_\alpha^{2m-3} c_\alpha d_\alpha,\ s_\alpha^{2m-5} c_\alpha d_\alpha,\ \ldots,\ s_\alpha c_\alpha d_\alpha \\ \alpha = 1,\ 2,\ 3,\ \ldots,\ 2m-1 \end{vmatrix}$$

$$= (-1)^m \begin{vmatrix} s_\alpha^{2m},\ s_\alpha^{2m-2},\ \ldots,\ s_\alpha^2,\ s_\alpha^{2m-3} c_\alpha d_\alpha,\ s_\alpha^{2m-5} c_\alpha d_\alpha,\ \ldots,\ s_\alpha c_\alpha d_\alpha \\ \alpha = 1,\ 2,\ 3,\ \ldots,\ 2m-1 \end{vmatrix},$$

$$p(1)\begin{vmatrix} s_\alpha^{2m-2},\ s_\alpha^{2m-4},\ \dots,\ s_\alpha^2,\ 1,\ s_\alpha^{2m-3}c_\alpha d_\alpha,\ s_\alpha^{2m-5}c_\alpha d_\alpha,\ \dots,\ s_\alpha c_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2m-1 \end{vmatrix}$$

$$=\begin{vmatrix} 1, & 1, & \dots, & 1, & 1, & 0, & 0, & \dots, & 0 \\ s_1^{2m}, & s_1^{2m-2}, & \dots, & s_1^2, & 1, & s_1^{2m-3}c_1d_1, & s_1^{2m-5}c_1d_1, & \dots, & s_1c_1d_1 \\ s_2^{2m}, & s_2^{2m-2}, & \dots, & s_2^2, & 1, & s_2^{2m-3}c_2d_2, & s_2^{2m-5}c_2d_2, & \dots, & s_2c_2d_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{2m-1}^{2m}, & s_{2m-1}^{2m-2}, & \dots, & s_{2m-1}^2, & 1, & s_{2m-1}^{2m-3}c_{2m-1}d_{2m-1}, & s_{2m-1}^{2m-5}c_{2m-1}d_{2m-1}, & \dots, & s_{2m-1}c_{2m-1}d_{2m-1} \end{vmatrix}$$

$$=\begin{vmatrix} s_\alpha^{2m-2}c_\alpha^2,\ s_\alpha^{2m-4}c_\alpha^2,\ \dots,\ s_\alpha^2c_\alpha^2,\ c_\alpha^2,\ s_\alpha^{2m-3}c_\alpha d_\alpha,\ s_\alpha^{2m-5}c_\alpha d_\alpha,\ \dots,\ s_\alpha c_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2m-1 \end{vmatrix},$$

$$k^{2m}p\left(\frac{1}{k^2}\right)\begin{vmatrix} s_\alpha^{2m-2},\ s_\alpha^{2m-4},\ \dots,\ s_\alpha^2,\ 1,\ s_\alpha^{2m-3}c_\alpha d_\alpha,\ s_\alpha^{2m-5}c_\alpha d_\alpha,\ \dots,\ s_\alpha c_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2m-1 \end{vmatrix}$$

$$=\begin{vmatrix} 1, & k^2, & k^4, & \dots, & k^{2m-2}, & k^{2m}, & 0, & 0, & \dots, & 0 \\ s_1^{2m}, & s_1^{2m-2}, & s_1^{2m-4}, & \dots, & s_1^2, & 1, & s_1^{2m-3}c_1d_1, & s_1^{2m-5}c_1d_1, & \dots, & s_1c_1d_1 \\ s_2^{2m}, & s_2^{2m-2}, & s_2^{2m-4}, & \dots, & s_2^2, & 1, & s_2^{2m-3}c_2d_2, & s_2^{2m-5}c_2d_2, & \dots, & s_2c_2d_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{2m-1}^{2m}, & s_{2m-1}^{2m-2}, & s_{2m-1}^{2m-4}, & \dots, & s_{2m-1}^2, & 1, & s_{2m-1}^{2m-3}c_{2m-1}d_{2m-1}, & s_{2m-1}^{2m-5}c_{2m-1}d_{2m-1}, & \dots, & s_{2m-1}c_{2m-1}d_{2m-1} \end{vmatrix}$$

$$=\begin{vmatrix} s_\alpha^{2m-2}d_\alpha^2,\ s_\alpha^{2m-4}d_\alpha^2,\ \dots,\ s_\alpha^2d_\alpha^2,\ d_\alpha^2,\ s_\alpha^{2m-3}c_\alpha d_\alpha,\ s_\alpha^{2m-5}c_\alpha d_\alpha,\ \dots,\ s_\alpha c_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2m-1 \end{vmatrix};$$

and (33), (34) and (35) become

(40) $\operatorname{sn}(u_1+u_2+u_3+\dots+u_{2m-1})$

$$=\pm\frac{\begin{vmatrix} s_\alpha^{2m-1},\ s_\alpha^{2m-3},\ \dots,\ s_\alpha,\ s_\alpha^{2m-4}c_\alpha d_\alpha,\ s_\alpha^{2m-6}c_\alpha d_\alpha,\ \dots,\ c_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2m-1 \end{vmatrix}}{\begin{vmatrix} s_\alpha^{2m-2},\ s_\alpha^{2m-4},\ \dots,\ 1,\ s_\alpha^{2m-3}c_\alpha d_\alpha,\ s_\alpha^{m-5}c_\alpha d_\alpha,\ \dots,\ s_\alpha c_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2m-1 \end{vmatrix}},$$

(41) $\operatorname{cn}(u_1+u_2+u_3+\dots+u_{2m-1})$

$$=\pm\frac{\begin{vmatrix} s_\alpha^{2m-2}c_\alpha,\ s_\alpha^{2m-4}c_\alpha,\ \dots,\ c_\alpha,\ s_\alpha^{2m-3}d_\alpha,\ s_\alpha^{2m-5}d_\alpha,\ \dots,\ s_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2m-1 \end{vmatrix}}{\begin{vmatrix} s_\alpha^{2m-2},\ s_\alpha^{2m-4},\ \dots,\ 1,\ s_\alpha^{2m-3}c_\alpha d_\alpha,\ s_\alpha^{2m-5}c_\alpha d_\alpha,\ \dots,\ s_\alpha c_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2m-1 \end{vmatrix}},$$

(42) $\operatorname{dn}(u_1+u_2+u_3+\dots+u_{2m-1})$

$$=\pm\frac{\begin{vmatrix} s_\alpha^{2m-2}d_\alpha,\ s_\alpha^{2m-4}d_\alpha,\ \dots,\ d_\alpha,\ s_\alpha^{2m-3}c_\alpha,\ s_\alpha^{2m-5}c_\alpha,\ \dots,\ s_\alpha c_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2m-1 \end{vmatrix}}{\begin{vmatrix} s_\alpha^{2m-2},\ s_\alpha^{2m-4},\ \dots,\ 1,\ s_\alpha^{2m-3}c_\alpha d_\alpha,\ s_\alpha^{2m-5}c_\alpha d_\alpha,\ \dots,\ s_\alpha c_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2m-1 \end{vmatrix}}.$$

If $$u_{2m-1}=0,\ s_{2m-1}=0,\ c_{2m-1}=1,\ d_{2m-1}=1,$$
(40), (41) and (42) become, on putting $m-1=n$,

(43) $$\mathrm{sn}(u_1+u_2+u_3+\dots+u_{2n}) = \pm\frac{\begin{vmatrix} s_\alpha^{2n},\ s_\alpha^{2n-2},\ \dots,\ 1,\ s_\alpha^{2n-3}c_\alpha d_\alpha,\ s_\alpha^{2n-5}c_\alpha d_\alpha,\ \dots,\ s_\alpha c_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2n \end{vmatrix}}{\begin{vmatrix} s_\alpha^{2n-1},\ s_\alpha^{2n-3},\ \dots,\ s_\alpha,\ s_\alpha^{2n-2}c_\alpha d_\alpha,\ s_\alpha^{2n-4}c_\alpha d_\alpha,\ \dots,\ c_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2n \end{vmatrix}},$$

(44) $$\mathrm{cn}(u_1+u_2+u_3+\dots,+u_{2n}) = \pm\frac{\begin{vmatrix} s_\alpha^{2n-1}c_\alpha,\ s_\alpha^{2n-3}c_\alpha,\ \dots,\ s_\alpha c_\alpha,\ s_\alpha^{2n-2}d_\alpha,\ s_\alpha^{2n-4}d_\alpha,\ \dots,\ d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2n \end{vmatrix}}{\begin{vmatrix} s_\alpha^{2n-1},\ s_\alpha^{2n-3},\ \dots,\ s_\alpha,\ s_\alpha^{2n-2}c_\alpha d_\alpha,\ s_\alpha^{2n-4}c_\alpha d_\alpha,\ \dots,\ c_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2n \end{vmatrix}},$$

(45) $$\mathrm{dn}(u_1+u_2+u_3+\dots,+u_{2n}) = \pm\frac{\begin{vmatrix} s_\alpha^{2n-1}d_\alpha,\ s_\alpha^{2n-3}d_\alpha,\ \dots,\ s_\alpha d_\alpha,\ s_\alpha^{2n-2}c_\alpha,\ s_\alpha^{2n-4}c_\alpha,\ \dots,\ c_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2n \end{vmatrix}}{\begin{vmatrix} s_\alpha^{2n-1},\ s_\alpha^{2n-3},\ \dots,\ s_\alpha,\ s_\alpha^{2n-2}c_\alpha d_\alpha,\ s_\alpha^{2n-4}c_\alpha d_\alpha,\ \dots,\ c_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2n \end{vmatrix}},$$

which are then the addition formulae for an even number of arguments.

Formulae (40)–(42) give, with a choice of signs which is at present arbitrary,

(46) $$\mathrm{sn}(u_1+u_2+u_3+\dots+u_{2n+1}) = (-1)^n\frac{\begin{vmatrix} s_\alpha^{2n+1},\ s_\alpha^{2n-1},\ \dots,\ s_\alpha,\ s_\alpha^{2n-2}c_\alpha d_\alpha,\ s_\alpha^{2n-4}c_\alpha d_\alpha,\ \dots,\ c_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2n+1 \end{vmatrix}}{\begin{vmatrix} s_\alpha^{2n},\ s_\alpha^{2n-2},\ \dots,\ 1,\ s_\alpha^{2n-1}c_\alpha d_\alpha,\ s_\alpha^{2n-3}c_\alpha d_\alpha,\dots,\ s_\alpha c_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2n+1 \end{vmatrix}},$$

(47) $$\mathrm{cn}(u_1+u_2+u_3+\dots+u_{2n+1}) = \frac{\begin{vmatrix} s_\alpha^{2n}c_\alpha,\ s_\alpha^{2n-2}c_\alpha,\ \dots,\ c_\alpha,\ s_\alpha^{2n-1}d_\alpha,\ s_\alpha^{2n-3}d_\alpha,\ \dots,\ s_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2n+1 \end{vmatrix}}{\begin{vmatrix} s_\alpha^{2n},\ s_\alpha^{2n-2},\ \dots,\ 1,\ s_\alpha^{2n-1}c_\alpha d_\alpha,\ s_\alpha^{2n-3}c_\alpha d_\alpha,\ \dots,\ s_\alpha c_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2n+1 \end{vmatrix}},$$

(48) $$\mathrm{dn}(u_1+u_2+u_3+\dots+u_{2n+1}) = \frac{\begin{vmatrix} s_\alpha^{2n}d_\alpha,\ s_\alpha^{2n-2}d_\alpha,\ \dots,\ d_\alpha,\ s_\alpha^{2n-1}c_\alpha,\ s_\alpha^{2n-3}c_\alpha,\ \dots,\ s_\alpha c_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2n+1 \end{vmatrix}}{\begin{vmatrix} s_\alpha^{2n},\ s_\alpha^{2n-2},\ \dots,\ 1,\ s_\alpha^{2n-1}c_\alpha d_\alpha,\ s_\alpha^{2n-3}c_\alpha d_\alpha,\ \dots,\ s_\alpha c_\alpha d_\alpha \\ \alpha=1,\ 2,\ 3,\ \dots,\ 2n+1 \end{vmatrix}},$$

from which we get, on putting $u_{2n+1} = 0$ and dividing numerators and denominators by $s_1 s_2 s_3 \ldots s_{2n}$,

(49) $\operatorname{sn}(u_1 + u_2 + u_3 + \ldots + u_{2n})$

$$= \frac{\begin{vmatrix} s_\alpha^{2n},\ s_\alpha^{2n-2},\ \ldots,\ 1,\ s_\alpha^{2n-3} c_\alpha d_\alpha,\ s_\alpha^{2n-5} c_\alpha d_\alpha,\ \ldots,\ s_\alpha c_\alpha d_\alpha \\ \alpha = 1,\ 2,\ 3,\ \ldots,\ 2n \end{vmatrix}}{\begin{vmatrix} s_\alpha^{2n-1},\ s_\alpha^{2n-3},\ \ldots,\ s_\alpha,\ s_\alpha^{2n-2} c_\alpha d_\alpha,\ s_\alpha^{2n-4} c_\alpha d_\alpha,\ \ldots,\ c_\alpha d_\alpha \\ \alpha = 1,\ 2,\ 3,\ \ldots,\ 2n \end{vmatrix}},$$

(50) $\operatorname{cn}(u_1 + u_2 + u_3 + \ldots + u_{2n})$

$$= \frac{\begin{vmatrix} s_\alpha^{2n-1} c_\alpha,\ s_\alpha^{2n-3} c_\alpha,\ \ldots,\ s_\alpha c_\alpha,\ s_\alpha^{2n-2} d_\alpha,\ s_\alpha^{2n-4} d_\alpha,\ \ldots,\ d_\alpha \\ \alpha = 1,\ 2,\ 3,\ \ldots,\ 2n \end{vmatrix}}{\begin{vmatrix} s_\alpha^{2n-1},\ s_\alpha^{2n-3},\ \ldots,\ s_\alpha,\ s_\alpha^{2n-2} c_\alpha d_\alpha,\ s_\alpha^{2n-4} c_\alpha d_\alpha,\ \ldots,\ c_\alpha d_\alpha \\ \alpha = 1,\ 2,\ 3,\ \ldots,\ 2n \end{vmatrix}},$$

(51) $\operatorname{dn}(u_1 + u_2 + u_3 + \ldots + u_{2n})$

$$= \frac{\begin{vmatrix} s_\alpha^{2n-1} d_\alpha,\ s_\alpha^{2n-3} d_\alpha,\ \ldots,\ s_\alpha d_\alpha,\ s_\alpha^{2n-2} c_\alpha,\ s_\alpha^{2n-4} c_\alpha,\ \ldots,\ c_\alpha \\ \alpha = 1,\ 2,\ 3,\ \ldots,\ 2n \end{vmatrix}}{\begin{vmatrix} s_\alpha^{2n-1},\ s_\alpha^{2n-3},\ \ldots,\ s_\alpha,\ s_\alpha^{2n-2} c_\alpha d_\alpha,\ s_\alpha^{2n-4} c_\alpha d_\alpha,\ \ldots,\ c_\alpha d_\alpha \\ \alpha = 1,\ 2,\ 3,\ \ldots,\ 2n \end{vmatrix}},$$

and from these again we obtain, on putting $u_{2n} = 0$ and dividing numerators and denominators by $s_1 s_2 s_3 \ldots s_{2n-1}$,

$\operatorname{sn}(u_1 + u_2 + u_3 + \ldots + u_{2n-1})$

$$= (-1)^{n-1} \frac{\begin{vmatrix} s_\alpha^{2n-1},\ s_\alpha^{2n-3},\ \ldots,\ s_\alpha,\ s_\alpha^{2n-4} c_\alpha d_\alpha,\ s_\alpha^{2n-6} c_\alpha d_\alpha,\ \ldots,\ c_\alpha d_\alpha \\ \alpha = 1,\ 2,\ 3,\ \ldots,\ 2n-1 \end{vmatrix}}{\begin{vmatrix} s_\alpha^{2n-2},\ s_\alpha^{2n-4},\ \ldots,\ 1,\ s_\alpha^{2n-3} c_\alpha d_\alpha,\ s_\alpha^{2n-5} c_\alpha d_\alpha,\ \ldots,\ s_\alpha c_\alpha d_\alpha \\ \alpha = 1,\ 2,\ 3,\ \ldots,\ 2n-1 \end{vmatrix}},$$

$\operatorname{cn}(u_1 + u_2 + u_3 + \ldots + u_{2n-1}$

$$= \frac{\begin{vmatrix} s_\alpha^{2n-2} c_\alpha,\ s_\alpha^{2n-4} c_\alpha,\ \ldots,\ c_\alpha,\ s_\alpha^{2n-3} d_\alpha,\ s_\alpha^{2n-5} d_\alpha,\ \ldots,\ s_\alpha d_\alpha \\ \alpha = 1,\ 2,\ 3,\ \ldots,\ 2n-1 \end{vmatrix}}{\begin{vmatrix} s_\alpha^{2n-2},\ s_\alpha^{2n-4},\ \ldots,\ 1,\ s_\alpha^{2n-3} c_\alpha d_\alpha,\ s_\alpha^{2n-5} c_\alpha d_\alpha,\ \ldots,\ s_\alpha c_\alpha d_\alpha \\ \alpha = 1,\ 2,\ 3,\ \ldots,\ 2n-1 \end{vmatrix}},$$

$\operatorname{dn}(u_1 + u_2 + u_3 + \ldots + u_{2n-1})$

$$= \frac{\begin{vmatrix} s_\alpha^{2n-2} d_\alpha,\ s_\alpha^{2n-4} d_\alpha,\ \ldots,\ d_\alpha,\ s_\alpha^{2n-3} c_\alpha,\ s_\alpha^{2n-5} c_\alpha,\ \ldots,\ s_\alpha c_\alpha \\ \alpha = 1,\ 2,\ 3,\ \ldots,\ 2n-1 \end{vmatrix}}{\begin{vmatrix} s_\alpha^{2n-2},\ s_\alpha^{2n-4},\ \ldots,\ 1,\ s_\alpha^{2n-3} c_\alpha d_\alpha,\ s_\alpha^{2n-5} c_\alpha d_\alpha,\ \ldots,\ s_\alpha c_\alpha d_\alpha \\ \alpha = 1,\ 2,\ 3,\ \ldots,\ 2n-1 \end{vmatrix}}.$$

From these formulæ it appears that, in passing from the sum of an odd number $2n - 1$ of arguments to the sum of the next even number $2n$ of arguments, the sign in the formula for sn does or does not change according as $n - 1$ is odd or even, but the signs of cn and dn remain unchanged, an odd or even value of $n - 1$; while, in passing from the sum of an even number $2n$ of arguments to the sum of the next odd number $2n + 1$ of arguments, the sign in the formula for sn does or does not change according as n is odd or even, and the signs of cn and dn remain unchanged for an odd or even value of n; *i. e.* in the formula for sn of the sum of an even number of arguments, and in those for cn and dn for the sum of any number of arguments, the sign is invariable, while in the formula for sn of the sum of the successive odd numbers of arguments the sign is alternately $+$ and $-$. But for the sum of two arguments the signs in the formulæ for sn, cn and dn are evidently all $+$, therefore (46)–(51) are correct even to their signs.

It remains only to point out the modifications of the formulæ which are necessary when several arguments are equal, in accordance with the principles above established. It is evident that, if $u_\alpha = u_{\alpha+1} = u_{\alpha+2} = \ldots = u_{\alpha+\mu-1}$, the $\alpha + 1^{\text{th}}$, $\alpha + 2^{\text{th}}$, $\ldots$ $\alpha + \mu - 1^{\text{th}}$ rows of the numerator and denominator of the right member of each of the formulæ (46)–(51) has to be replaced respectively by the 1^{st}, 2^{nd}, $\ldots$, $\mu - 1^{\text{th}}$ derivatives of the α^{th} row of that numerator or denominator. As any common factor of numerator and denominator will disappear from the quotient, it is evident that the derivatives may be taken with respect to u_α instead of s_α, as above, and we know that

$$\frac{\partial s_\alpha}{\partial u_\alpha} = c_\alpha d_\alpha, \quad \frac{\partial c_\alpha}{\partial u_\alpha} = - s_\alpha d_\alpha, \quad \frac{\partial d_\alpha}{\partial u_\alpha} = - k^2 s_\alpha c_\alpha,$$

which enables us to determine the $\alpha + 1^{\text{th}}$, $\alpha + 2^{\text{th}}$, $\ldots$, $\alpha + \mu - 1^{\text{th}}$ row in the case supposed, but the general formulæ seem too complicated to be useful.

BALTIMORE, June 15, 1885.